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Technical Note

Forced convection in a parallel plate channel with asymmetric heating

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Abstract

A Nusselt number, appropriate for forced convection in a channel bounded by two parallel plate walls heated asymmetrically, is introduced and evaluated for various velocity profiles, for either uniform heat flux or uniform temperature boundary conditions. It is shown that the value of this new Nusselt number is independent of the asymmetry if and only if the velocity profile is symmetric with respect to the midline of the channel.

1. Introduction

The problem of forced convection in a channel between two parallel plate walls is a classical problem that has been revisited in recent years in connection with the cooling of electronic equipment using materials involving hyperporous media or microchannels. Recently published textbooks and handbooks, such as those by Bejan [1] and Kakaç et al. [2], devote substantial space to the case of symmetric heating but little to the more complicated case of asymmetric heating. However, this case is mentioned in Shah and London [3, pp. 155–157] and Kakaç et al. [2, pp. 3.31–3.32], where the following results are given, without details of derivation. (An outline derivation is given in Kays and Crawford [4].)

$$Nu_1 \equiv \frac{2Hq_1''}{k(T_{\mathrm{w1}}-T_{\mathrm{m}})} = \frac{140}{26-9(q_2''/q_1'')}$$
$$Nu_2 \equiv \frac{2Hq_2''}{k(T_{\mathrm{w2}}-T_{\mathrm{m}})} = \frac{140}{26-9(q_1''/q_2'')} \qquad (1a,b)$$

Here Nu_1 and Nu_2 are Nusselt numbers defined separately for the two walls. These numbers are defined in terms of the hydraulic diameter (twice H, where H is the distance between the plates), the constant heat fluxes q_1'' and q_2'' directed into the channel at the two walls, and the wall temperatures T_{w1} and T_{w2}, while T_{m} is the mixing cup mean temperature and k the thermal conductivity.

It is obvious that Nu_1 becomes infinite and then negative when q_2''/q_1'' increases through the value 26/9. Of course, this simply corresponds to the fact that the temperature difference $T_{\mathrm{w1}}-T_{\mathrm{m}}$ then passes though a zero value. Nevertheless, the presence of such singularities reduces the usefulness of the Nusselt numbers defined in this way, and this suggests that there might be a better way of characterizing the heat transfer.

An influx of q_1'' and q_2'' at the respective walls is equivalent to the superposition of heat fluxes $(q_1''-q_2'')/2$ into the channel at each wall and a through flux $(q_1''+q_2'')/2$ across the channel from the first wall to the second. Forced convection essentially involves a balance between the net influx of heat into the channel and convective transport along the channel. The through flux plays a passive role. This fact, together with the fact that the

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thermal energy equation (Eq. (4) below) is linear in the temperature variable, suggests that there might be circumstances in which it is sufficient to concentrate on a study of the effect of the net influx. Indeed there are such circumstances.

In this paper the following theorem is established, first for the case where the heat flux at each boundary wall is uniform along the wall and then for the case where the temperature at each wall is held uniform: *The value of the Nusselt number defined in terms of the average heat flux and the average wall temperature is independent of the ratio* q_1''/q_2'' *(and so is identical with value of the usual Nusselt number defined for the case of symmetric heating) provided that the velocity profile is symmetric with respect to the midline of the channel.*

Clearly, this theorem does not apply to Couette flow, the subject of the study by Xiong and Kuznetsov [5]. However, the result is relevant in cases that include Poiseuille flow and slug flow, and it means that a great deal of work on forced convection with symmetric heating (such as that reported in Nield et al. [6] and in studies referenced in that paper or reviewed by Nield and Bejan [7]) can, without further calculation, be applied in a wider context, to asymmetric heating as well as symmetric heating. This is a major gain, because so far few studies of the asymmetric heating situation have been published.

2. Analysis

We consider fully developed convective flow. The situation considered is shown in Fig. 1. The dimensional variables are denoted by means of asterisks. The average wall temperature and average wall heat flux are defined as

$$T_{w\mu} = \frac{1}{2}(T_{w1} + T_{w2}) \qquad q''_\mu = \frac{1}{2}(q''_1 + q''_2) \tag{2a,b}$$

The mean velocity and mean temperature are defined by

$$u_m = \frac{1}{H}\int_0^H u^*\,dy^* \qquad T_m = \frac{1}{u_m H}\int_0^H u^* T^*\,dy^* \tag{3a,b}$$

The thermal energy equation, for the case of large Péclet number so that the longitudinal conduction is negligible, is

$$u^*\frac{\partial T^*}{\partial x^*} = \frac{k}{\rho c_P}\frac{\partial^2 T^*}{\partial y^{*2}} \tag{4}$$

The first law of thermodynamics implies that

$$\frac{dT^*_m}{dx^*} = \frac{2q''_\mu}{\rho c_P H u_m} \tag{5}$$

We now introduce nondimensional variables defined by

$$u = \frac{u^*}{u_m}, \quad T = \frac{T^* - T_{w\mu}}{T_m - T_{w\mu}}, \quad y = \frac{y^*}{H} \tag{6a,b,c}$$

and define a Nusselt number by

$$Nu = \frac{2Hq''_\mu}{k(T_{w\mu} - T_m)} \tag{7}$$

For the isoflux case, Eqs. (4) through (7) then lead to

$$\frac{d^2T}{dy^2} = -Nu\ u \tag{8}$$

The boundary conditions lead to

$$T(0) = \beta, \quad T(1) = -\beta \tag{9a,b}$$

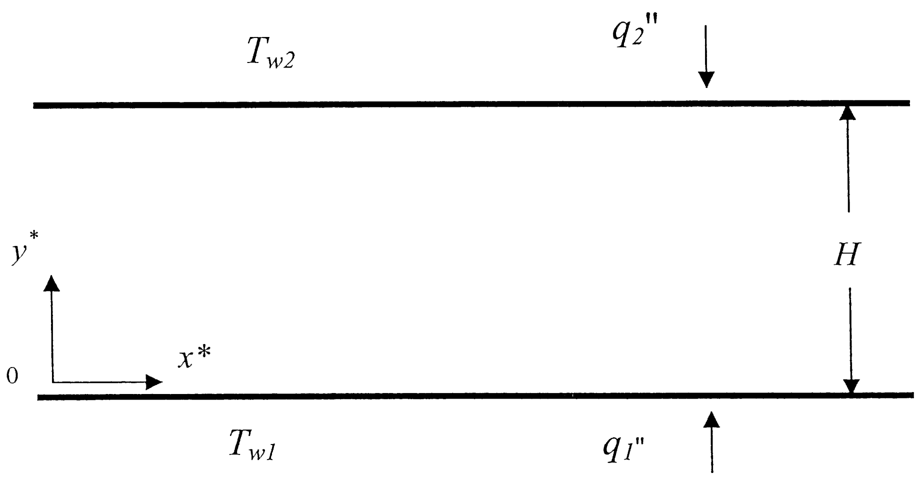

Fig. 1. Definition sketch.

where, for convenience, we have introduced the shorthand notation

$$\beta = \frac{T_{w1} - T_{w2}}{2(T_m - T_{w\mu})} \tag{10}$$

We now consider velocity profiles of the form

$$u = c_0 + c_1 y + c_2 y^2 \tag{11}$$

Particular special cases are

(i) *Slug flow:* $c_0 = 1$, $c_1 = 0$, $c_2 = 0$;
(ii) *Poiseuille flow:* $c_0 = 0$, $c_1 = 6$, $c_2 = -6$;
(iii) *Couette flow:* $c_0 = 0$, $c_1 = 2$, $c_2 = 0$.

The solution of Eq. (8) subject to the boundary conditions (9a,b) is then

$$T = \frac{1}{12} Nu\{6c_0(y - y^2) + 2c_1(y - y^3) + c_2(y - y^4)\} + \beta(1 - 2y) \tag{12}$$

The boundary heat fluxes can now be computed as

$$q_1'' = q_\mu''\left\{-\frac{4\beta}{Nu} + \alpha_1\right\}$$
$$q_2'' = q_\mu''\left\{\frac{4\beta}{Nu} + \alpha_2\right\} \tag{13a,b}$$

where $\alpha_1 = \alpha_2 = 1$ for either slug flow or Poiseuille flow, and $\alpha_1 = 2/3$, $\alpha_2 = 4/3$ for the case of Couette flow. It follows that

$$\beta = Nu\left\{\frac{q_2'' - q_1''}{4(q_1'' + q_2'')} + \frac{\alpha_1 - \alpha_2}{8}\right\} \tag{14}$$

Substitution of u and T into the compatibility condition

$$\int_0^1 uT\,dy = 1 \tag{15}$$

yields the value of the Nusselt number. The following results are readily obtained.

$$\text{Slug flow}: \quad Nu = 12 \tag{16a}$$

$$\text{Poiseuille flow}: \quad Nu = 140/17 \tag{16b}$$

$$\text{Couette flow}: \quad Nu = (45 + 15\beta)/4 \tag{16c}$$

It is remarkable that, in the case of either slug flow or Poiseuille flow, the value of Nu is independent of β, and hence independent of whether the heating is symmetric or asymmetric. In the case of Couette flow the use of Eq. (14), together with Eq. (16c), leads to the expression

$$Nu = \frac{30q_1'' + 30q_2''}{6q_1'' + q_2''} \tag{17}$$

To complete the picture, we note that some simple algebra leads to the relationships between our Nusselt number Nu and the Nusselt numbers Nu_1 and Nu_2 employed by Shah and London [3], as given by Eq. (1a,b), for the case of asymmetric flow.

In the case of *slug flow*, one finds that

$$Nu_1 = \frac{12}{2 - (q_2''/q_1'')}$$
$$Nu_2 = \frac{12}{2 - (q_1''/q_2'')} \tag{18a,b}$$

In the case *of Poiseuille flow* one obtains the expressions given in Eq. (1a,b), as expected.

In the case of *Couette flow,* a more complicated calculation gives

$$Nu_1 = \frac{15 + 15(q_2''/q_1'')}{8 + 6(q_2''/q_1'') - 2(q_2''/q_1'')^2}$$
$$Nu_2 = \frac{15 + 15(q_1''/q_2'')}{3 + (q_1''/q_2'') - 2(q_1''/q_2'')^2} \tag{19a,b}$$

The results in Eq. (18a,b) agree with those in [3], while those in Eq. (19a,b) are thought to be new. It is clearly preferable to use Nu rather than Nu_1 and Nu_2, on the grounds of simplicity as well as absence of any singularity.

3. Proof of the theorem for the isoflux case

We have already shown that Nu is independent of β in the case of slug flow and Poiseuille flow. We now show that the same is true more generally, for any velocity profile that is symmetric about the midline of the channel. In terms of the transformed variable $y' = y - 1/2$, we then have $u = f_1(y')$, an even function of y', and since the only derivative in the transformed form of Eq. (8) is of even order, it follows that the solution is of the form

$$T = Nu f_2(y') - 2\beta y' \tag{20}$$

and so

$$uT = Nu f_3(y') + \beta g(y') \tag{21}$$

where $f_2(y')$ and $f_3(y')$ are even functions and $g(y')$ is an odd function. When we substitute in the compatibility equation

$$\int_{-1/2}^{1/2} uT\,dy' = 1 \tag{22}$$

we find that the term in β vanishes, because of cancellation, and we are left with

$$Nu = \left\{\int_{-1/2}^{1/2} f_3(y')\,dy'\right\}^{-1} \tag{23}$$

and this is obviously independent of β, and hence independent of whether the heating is symmetric or asymmetric.

4. Proof of the theorem for the isotemperature case, and discussion

For the isoflux case (heat fluxes held uniform at each wall) it has been established that, when the Nusselt number is defined in terms of the average wall heat flux and average temperature, the value of this Nusselt number does not depend on whether the walls are heated symmetrically or asymmetrically, provided that the velocity profile is symmetric about the center line of the channel. This means that many results reported in the literature for the restricted case of symmetric heating are also applicable in the more general case of asymmetric heating.

In the isotemperature case (uniform wall temperature at each wall) the above argument involving even and odd functions is still valid in its essential features, but the details of the proof are complicated in the general case because now the solution for T emerges as an eigenfunction in a problem that involves Nu as an eigenvalue. However, T is still expressible in the form

$$T = Nuf(y') + \beta g(y') \tag{24}$$

where $f(y')$ is an even function and $g(y')$ is an odd function, and the rest of the proof follows through as before. For the special case of slug flow ($u = 1$) the details of the proof can be filled in easily. The problem reduces to

$$\begin{aligned} &\frac{d^2T}{dy'^2} = -Nu\,uT \\ &T(-\frac{1}{2}) = \beta, \quad T(\frac{1}{2}) = -\beta \end{aligned} \tag{25a,b}$$

and the relevant solution of this eigenvalue problem (and the compatibility condition) is given by

$$\begin{aligned} &Nu = \pi^2 \\ &T = \frac{\pi}{2}\cos \pi y' - \beta \sin \pi y' \end{aligned} \tag{26a,b}$$

We conclude with a general remark. As far as the author is aware, there has been no previously published work on the isothermal asymmetric heating case for forced convection in a channel, and a likely reason for this can now be suggested. If one starts by thinking in terms of a Nusselt number for each wall, then it is impossible to follow the argument given in Section 3.4.4 of Bejan [1], because one obtains two temperature differences that decay exponentially in the direction of flow at different rates, and this is obviously inconsistent with the assumption of fully developed convection. When one applies the argument using the average wall heat flux and the average wall temperature then there is no difficulty.

5. Further discussion

For some practical purposes a knowledge of the average Nusselt number on its own may not provide sufficient information. For example, in the case of specified wall heat flux one might like to know something about the individual wall temperatures. Once Nu has been determined, the value of the parameter β can be calculated from Eq. (14), and with both Nu and β known the defining Eqs. (7) and (10) allow the two wall temperatures (relative to bulk temperature) to be found. In fact, from Eqs. (7) and (10) one obtains the formulas

$$\begin{aligned} &T_{w1} - T_m = \frac{Hq''}{Nu}(1 - 2\beta) \\ &T_{w2} - T_m = \frac{Hq''}{Nu}(1 + 2\beta) \end{aligned} \tag{27a,b}$$

In other words, the information that one might otherwise obtain from values Nu_1 and Nu_2 is available.

A similar situation applies to the isotemperature case. With the two wall temperatures (relative to the bulk temperature) specified, β is known from Eq. (10), and with both Nu and β known, Eqs. (7) and (13a,b) determine the two wall heat fluxes.

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